Standard Model

I. Dirac Equation & Antiparticles

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Schrödinger equation

- Consider **non-relativistic** particle of mass $\boldsymbol{m}$ in a potential U: $E = \frac{p^2}{2m} + U$
- Then substitute the energy and momentum operators: $\vec{p} \to -i\nabla,\, E \to i\frac{\partial}{\partial t}$
- What gives the **non-relativistic** Schrödinger equation: $\left(-\frac{\nabla^2}{2m} + U\right)\Psi = i\frac{\partial\Psi}{\partial t}$
- The solution for the free particle ($U = 0$): $\Psi(\vec{x},t) \propto e^{-iEt}\,\psi(\vec{x})$

The Schrödinger equation is 1st order in $\boldsymbol{\frac{\partial}{\partial t}}$ but second in $\boldsymbol{\frac{\partial}{\partial x}}$.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

- For a **relativistic particles** space and time should be treated equally.
- For a relativistic particle the energy-momentum relationship is: $E^2 - p^2 = m^2$ and covariant: $p^\mu p_\mu - m^2 = 0$
- Substituting the energy and momentum operators we have Klein-Gordon equation: $-\frac{\partial^2}{\partial t^2}\Psi + \nabla^2\Psi = m^2\Psi$

  also written in the Lorentz covariant way: $(-\partial^\mu\partial_\mu - m^2)\psi = 0$

- The free particle solutions are plane waves: $\Psi(\vec{x},t) \propto e^{-i(Et - \vec{p}\cdot\vec{x})}$
- With positive and negative energy solutions: $E = \pm\sqrt{p^2 - m^2}$

The Klein-Gordon equation is Lorentz invariant but gives the negative energy solutions. And describes only bosons.

Dirac equation

- Dirac (1928) formulated the alternative wave equation for relativistic particles as simple „square root" of the Klein-Gordon (K-G) equation: $\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \nabla - m\right)\psi = 0$

$$-\frac{\partial^2}{\partial t^2}\Psi + \nabla^2\Psi = m^2\Psi$$

  or in shorter notation: $\boxed{\boldsymbol{(i\gamma^\mu \partial_\mu - m)\psi = 0}}$ Dirac Equation in the covariant form

  where: $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are unknown coefficients to be defined….

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- To find how the $\boldsymbol{\gamma^\mu}$ should look like, first multiply the Dirac equation by its conjugate equation:

$$\psi^\dagger\left(-i\gamma^0 \frac{\partial}{\partial t} - i\vec{\gamma} \cdot \nabla - m\right)\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \nabla - m\right)\psi = 0$$

  what should restore the K-G equation.

- This leads to the conditions on the $\boldsymbol{\gamma^\mu}$:

  $(\gamma^0)^2 = 1$,

  $(\gamma^i)^2 = -1$, $i = 1, 2, 3$

  $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ for $\mu \neq \nu$,

  or in terms of anticommutation relation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Dirac equation - γ matrices

One of the solutions (among many others) for the $\boldsymbol{\gamma^\mu}$ are 4x4 unitary matrices (Dirac-Pauli representation):

$$\gamma^0 = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}} \end{pmatrix} \qquad \boldsymbol{I}, \quad -\boldsymbol{I}$$

$$\gamma^1 = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \qquad \boldsymbol{\sigma^1}, \quad -\boldsymbol{\sigma^1}$$

$$\gamma^2 = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & \boxed{\begin{matrix} 0 & -i \\ i & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & i \\ -i & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \qquad \boldsymbol{\sigma^2}, \quad -\boldsymbol{\sigma^2}$$

$$\gamma^3 = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}} \\ \boxed{\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \qquad \boldsymbol{\sigma^3}, \quad -\boldsymbol{\sigma^3}$$

$\boldsymbol{\gamma^\mu}$ are fixed matrices,

$\boldsymbol{\sigma^i}$: Pauli spin matrices (representation of the 1/2 spin operator)

The full version of **Dirac Equation (*DE*)**:

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

bi-spinor

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

remember about summation over repeated indices!

Dirac equation - γ matrices

One of the solutions (among many others) for the $\boldsymbol{\gamma^{\mu}}$ are 4x4 unitary matrices (Dirac-Pauli representation):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^{\mu} = \begin{pmatrix} 0 & \boldsymbol{\sigma^{\mu}} \\ -\boldsymbol{\sigma^{\mu}} & 0 \end{pmatrix}$$

every element of $\boldsymbol{\gamma^{\mu}}$ matrices stands for 2x2 matrix,
1 denotes 2x2 unit matrix,
0 represents 2x2 null matrix

$\boldsymbol{\gamma^{\mu}}$ are fixed matrices;
$\boldsymbol{\sigma^{i}}$: Pauli spin matrices (representation of the 1/2 spin operator)

The full version of **Dirac Equation (*DE*)**:

$$\begin{pmatrix} i\dfrac{\partial}{\partial t} - m & 0 & i\dfrac{\partial}{\partial z} & i\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} \\ 0 & i\dfrac{\partial}{\partial t} - m & i\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y} & -i\dfrac{\partial}{\partial z} \\ -i\dfrac{\partial}{\partial z} & -i\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} & -i\dfrac{\partial}{\partial t} - m & 0 \\ -i\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} & i\dfrac{\partial}{\partial z} & 0 & -i\dfrac{\partial}{\partial t} - m \end{pmatrix} \underbrace{\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}}_{\text{bi-spinor}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{(\boldsymbol{i\gamma^{\mu}\partial_{\mu} - m)\psi = 0}}$$

remember about summation
over repeated indices!

Dirac equation – solutions

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

It's nice to remember what we wanted to obtain: a wavefunction of a relativistic particle.

Let's write it now in the form of the plane wave and a Dirac spinor, $u(p^\mu)$:

$$\psi(x^\mu) = u(p^\mu)e^{-i(Et-\vec{p}\cdot\vec{x})}$$

$$p^\mu = (E,\vec{p})$$
$$p_\mu = (E,-\vec{p})$$
$$x^\mu = (t,\vec{x})$$
$$-ix_\mu p^\mu = -i(Et-\vec{p}\cdot\vec{x})$$

Substituting $\psi(x^\mu)$ into *DE*: $\left(\boldsymbol{\gamma^\mu p_\mu - m}\right)\boldsymbol{u(E,\vec{p}) = 0}$

- For a particle at rest, $\vec{p} = \mathbf{0}$, spatial derivatives are 0, *DE* is in the form: $\left(i\gamma^0\frac{\partial}{\partial t} - m\right)\psi = 0$

$$(\gamma^0 E - m)\psi = 0$$

what can be expressed as an eigenvalue problem for the spinors u:

$$\hat{E}u = \begin{pmatrix} m\boldsymbol{I} & 0 \\ 0 & -m\boldsymbol{I} \end{pmatrix} u$$

- The free-particle wavefunction is: $\psi = u(E,0)e^{-iEt}$

with the eigenspinors:

spin up ↑ spin down ↓

$$\underbrace{u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E\,=\,m} \quad \underbrace{u_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad u_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E\,=\,-m}$$

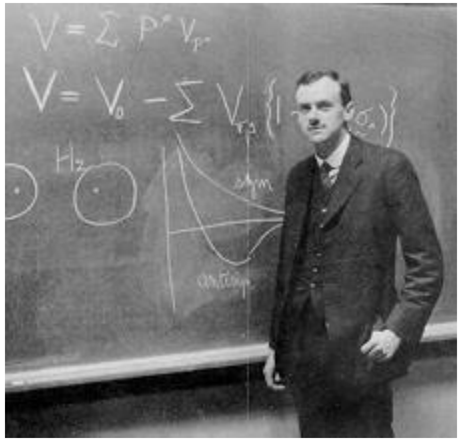
These four states are also eigenvalues of the $\hat{S}_z$ operator, so they represent spin-up and spin-down fermions (why? - later).

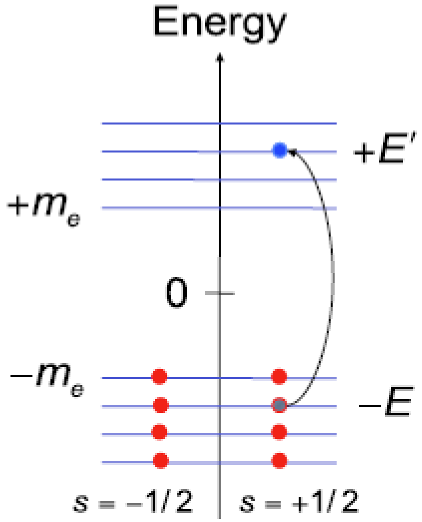
Dirac equation – interpretation I

Four solutions of the Dirac equation for a particle at rest:

$$\psi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imt} \qquad \psi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imt} \qquad \psi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{+imt} \qquad \psi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{+imt}$$

describe two different state of a fermion (↑↓) with $E = m$ and $E = -m$





Dirac's Interpretation:

- Vacuum (fully filled) represents a „sea" of negative energy particles.
- According to Dirac: holes in this „sea" represent antiparticles.
- If energy 2E is provided to the vacuum:

  one electron (negative charge, positive energy) and one hole (positive charge, negative energy are created.

- This picture fails for bosons!

Dirac equation – interpretation II

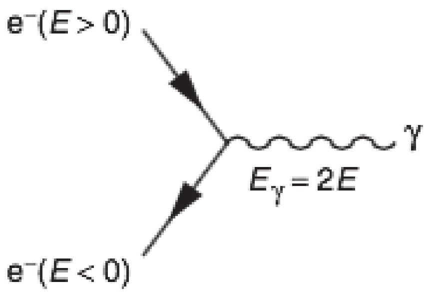
Stückelberg (1941)-Feynman (1948) interpretation of antiparticles*:

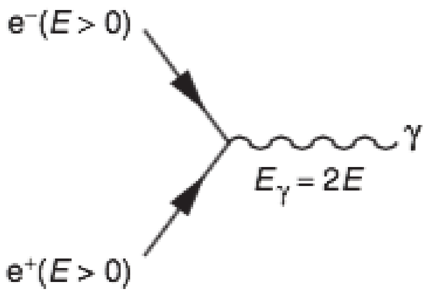
- consider the negative energy solution as *running backwards in time*

  and re-label it as *antiparticle*, with *positive* energy, going forward in time:

$$e^{-i[(-E)(-t)-(-\vec{p})\cdot(-\vec{x})]} = e^{-i[Et-\vec{p}\cdot\vec{x}]}$$

- emission of $E > 0$ antiparticle = absorption of particle $E < 0$



This involves a *CPT* transformation:

- we have flipped Charge (C),
- flipped time (T),
- *and to prevent momentum* from being flipped, must *also* flip the space coordinates (P)

***Feynman–Stueckelberg interpretation** [Wikipedia]
By considering the propagation of the negative energy modes of the electron field backward in time, Ernst Stueckelberg reached a pictorial understanding of the fact that the particle and antiparticle have equal mass **m** and spin **J** but opposite charges **q**. This allowed him to rewrite perturbation theory precisely in the form of diagrams. Richard Feynman later gave an independent systematic derivation of these diagrams from a particle formalism, and they are now called Feynman diagrams. Each line of a diagram represents a particle propagating either backward or forward in time. This technique is the most widespread method of computing amplitudes in quantum field theory today.
Since this picture was first developed by Stueckelberg,[5] and acquired its modern form in Feynman's work,[6] it is called the **Feynman–Stueckelberg interpretation** of antiparticles to honor both scientists.

Dirac equation – general solution

- For a moving particle, $\vec{p} \neq 0$, the Dirac equation can be written using Pauli representation in *DE*

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\psi(x^\mu) = u(p^\mu)e^{-i(Et-\vec{p}\cdot\vec{x})}$$

$$(\gamma^\mu p_\mu - m)u(E,\vec{p}) = 0$$

Tutorial!

using: $(i\gamma^\mu \partial_\mu - m)\psi = 0\,,\ \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$$(\gamma^\mu p_\mu - m)(u_A \quad u_B) = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$, where: $u_A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $u_B = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$

4-component bi-spinor as 2-component vector

- It seems that the equations for vectors u_A and u_B are coupled:

$$(\vec{\sigma}\cdot\vec{p})\, u_B = (E-m)\, u_A$$

$$(\vec{\sigma}\cdot\vec{p})\, u_A = (E+m)\, u_B$$

- Taking the two simplest solutions for u_A and u_B as the orthogonal vectors:

$$u_{A,B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } u_{A,B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Tutorial!

we obtain four orthogonal solutions to the free particle Dirac equation:

where:

$$\psi_i = u_i(E,\vec{p})e^{-i(Et-\vec{p}\cdot\vec{x})}$$

Dirac equation – solutions for moving particles

… where:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

electron with energy

$$E = +\sqrt{m^2 + p^2}$$

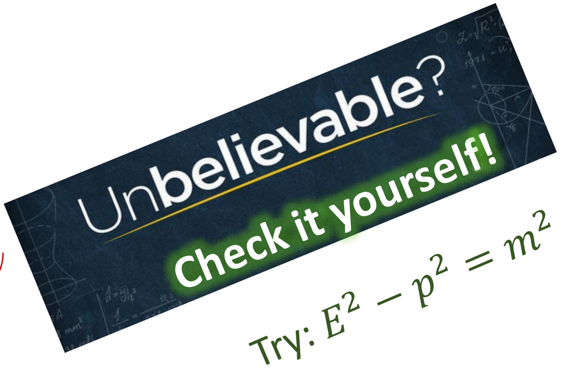
$$\psi = u_{1,2}(p^\mu)\, e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$u_3 = \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

positron with energy

$$E = -\sqrt{m^2 + p^2}$$

$$\psi = u_{3,4}(p^\mu)\, e^{-i(Et - \vec{p}\cdot\vec{x})} \qquad p^\mu = (E, \vec{p})$$



Try: $E^2 - p^2 = m^2$

Now we can take F-S interpretation of antiparticles as particles with positive energy (propagating backwards in time), and change the negative energy solutions $u_{3,4}$ to represent positive antiparticle (positron) spinors $v_{1,2}$:

$$v_1(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})} \equiv u_4(-E,-\vec{p})e^{-i(-Et + \vec{p}\cdot\vec{x})} = u_4(-E,-\vec{p})e^{i(Et - \vec{p}\cdot\vec{x})}$$

$$v_2(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})} \equiv u_3(-E,-\vec{p})e^{-i(-Et + \vec{p}\cdot\vec{x})} = u_3(-E,-\vec{p})e^{i(Et - \vec{p}\cdot\vec{x})}$$

$$E = +\sqrt{m^2 + p^2}$$

reversing the sign of *E* and *p*

The u and v are solutions of:

$$(i\gamma^\mu p_\mu - m)u = 0 \text{ and } (i\gamma^\mu p_\mu + m)v = 0$$

Dirac equation – solutions for moving particles

… where:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

electron with energy

$$E = +\sqrt{m^2 + p^2}$$

$$\psi = u_{1,2}(p^\mu)e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$v_2 = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

positron with energy

$$E = +\sqrt{m^2 + p^2}$$

$$\psi = v_{1,2}(p^\mu)\, e^{i(Et - \vec{p}\cdot\vec{x})}$$

$$p^\mu = (E, \vec{p})$$

The u and v are solutions of:

$$\boldsymbol{(i\gamma^\mu p_\mu - m)u = 0} \text{ and } \boldsymbol{(i\gamma^\mu p_\mu + m)v = 0}$$

Summary of Solutions of Dirac Equation

The normalised free particle solutions to the *DE*:

$\psi = u(E,\vec{p})\, e^{-i(Et-\vec{p}\cdot\vec{x})}$ satisfy $\left(i\gamma^\mu p_\mu - m\right)u = 0$

with:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

The normalised free antiparticle solutions to the *DE*:

$\psi = u(E,\vec{p})\, e^{i(Et+\vec{p}\cdot\vec{x})}$ satisfy $\left(i\gamma^\mu p_\mu + m\right)u = 0$

with:

$$v_1 = \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

- Both particle and antiparticle have positive energy solutions: $E = +\sqrt{m^2 + p^2}$.
- Particle and antiparticle have opposite spin: $\hat{S}^v = -\hat{S}$

but now:

The main problem is:
we have no antiparticles....

Antiparticles? Where?

No evidence for the original, “primordial” cosmic antimatter:

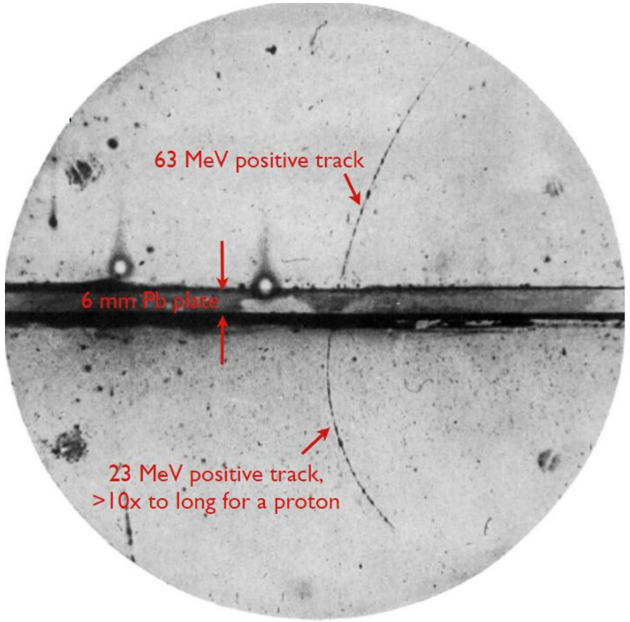
- Absence of anti-nuclei amongst cosmic rays in our galaxy
- Absence of intense γ−ray emission due to annihilation of distant galaxies in collision with antimatter

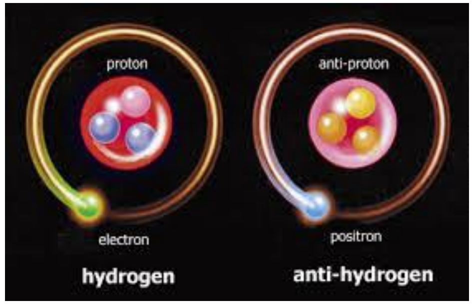
Antiparticles- history

1. **Positron** was discovered in 1933 by Anderson with usage of a Wilson cloud chamber.
2. **Antiproton** in 1955 at the Bevatron (a 6.5 GeV synchrotron in Berkeley)
3. **Antineutron** – 1956.

4. **ANTI-HYDRODEN**:
   Produced in 1995 at the Low Energy Antiproton Ring (LEAR) CERN

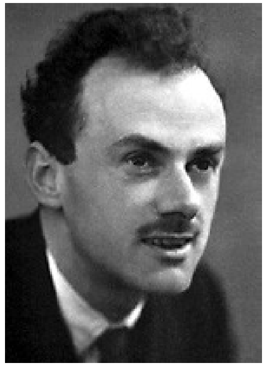
5. Searches of **anti-nuclei** in the space:
   - Alpha Magnetic Spectrometer (AMS-01)
   - Searches for anti-helium in cosmic rays – lots of He found but no anti-He!
   - AMS-02 – extreme flux of positrons detected – consistent with e+e- annihilation, but some increase in high energies of unknown origin.





Antiparticles- why?

Dirac's prescience



Concluding words of 1933 Nobel lecture

"If we accept the view of complete symmetry between positive and negative electric charge so far as concerns the fundamental laws of Nature, we must regard it rather as an accident that the Earth (and presumably the whole solar system), contains a preponderance of negative electrons and positive protons. It is quite possible that for some of the stars it is the other way about, these stars being built up mainly of positrons and negative protons. In fact, there may be half the stars of each kind. The two kinds of stars would both show exactly the same spectra, and there would be no way of distinguishing them by present astronomical methods."