CP Violation In Heavy Flavour Physics

I. Dirac Equation & Antiparticles

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Schrödinger equation

- Consider **non-relativistic** particle of mass $\boldsymbol{m}$ in a potential U: $E = \frac{p^2}{2m} + U$
- Then substitute the energy and momentum operators: $\vec{p} \to -i\nabla$, $E \to i\frac{\partial}{\partial t}$
- What gives the **non-relativistic** Schrödinger equation: $\left(-\frac{\nabla^2}{2m} + U\right)\Psi = i\frac{\partial\Psi}{\partial t}$
- The solution for the free particle ($U = 0$): $\Psi(\vec{x}, t) \propto e^{-iEt}\,\psi(\vec{x})$

The Schrödinger equation is 1st order in $\boldsymbol{\frac{\partial}{\partial t}}$ but second in $\boldsymbol{\frac{\partial}{\partial x}}$.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

- For a **relativistic particles** space and time should be treated equally.
- For a relativistic particle the energy-momentum relationship is: $E^2 - p^2 = m^2$ and covariant: $p^\mu p_\mu - m^2 = 0$
- Substituting the energy and momentum operators we have Klein-Gordon equation: $-\frac{\partial^2}{\partial t^2}\Psi + \nabla^2\Psi = m^2\Psi$

  also written in the Lorentz covariant way: $\left(-\partial^\mu\partial_\mu - m^2\right)\psi = 0$
- The free particle solutions are plane waves: $\Psi(\vec{x}, t) \propto e^{-i(Et - \vec{p}\cdot\vec{x})}$
- With positive and negative energy solutions: $E = \pm\sqrt{p^2 - m^2}$

The Klein-Gordon equation is Lorentz invariant but gives the negative energy solutions. And describes only bosons.

Dirac equation

- Dirac (1928) formulated the alternative wave equation for relativistic particles as simple „square root” of the Klein-Gordon (K-G) equation: $\left(i\gamma^0\frac{\partial}{\partial t}+i\vec{\gamma}\cdot\nabla-m\right)\psi=0$

$$-\frac{\partial^2}{\partial t^2}\Psi+\nabla^2\Psi=m^2\Psi$$

or in shorter notation:

$$\boxed{(\boldsymbol{i\gamma^\mu\partial_\mu-m})\boldsymbol{\psi}=\boldsymbol{0}}$$

Dirac Equation in the covariant form

where: $\gamma^\mu=(\gamma^0,\gamma^1,\gamma^2,\gamma^3)$ are unknown coefficients to be defined….

$$\partial_\mu\equiv\frac{\partial}{\partial x^\mu}=\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

- To find how the $\boldsymbol{\gamma^\mu}$ should look like, first multiply the Dirac equation by its conjugate equation:

$$\psi^\dagger\left(-i\gamma^0\frac{\partial}{\partial t}-i\vec{\gamma}\cdot\nabla-m\right)\left(i\gamma^0\frac{\partial}{\partial t}+i\vec{\gamma}\cdot\nabla-m\right)\psi=0$$

Tutorial!

what should restore the K-G equation.

- This leads to the conditions on the $\boldsymbol{\gamma^\mu}$:

$(\gamma^\mu)^2=1\ ,\ \mu,\nu=0,1,2,3$

$\left(\gamma^i\right)^2=-1\ ,\ i=1,2,3$

$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=0$ for $\mu\neq\nu,$

or in terms of anticommutation relation: $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

Dirac equation - γ matrices

One of the solutions (among many others) for the γ^μ are 4x4 unitary matrices (Dirac-Pauli representation):

I $\quad$ σ^1 $\quad$ σ^2 $\quad$ σ^3

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$-I$ $\quad$ $-\sigma^1$ $\quad$ $-\sigma^2$ $\quad$ $-\sigma^3$

γ^μ are fixed matrices,

σ^i: Pauli spin matrices (representation of the 1/2 spin operator)

The full version of **Dirac Equation (*DE*)**:

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial z} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

bi-spinor

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

remember about summation over repeated indices!

OMG CHECK IT OUT

Dirac equation - γ matrices

One of the solutions (among many others) for the $\boldsymbol{\gamma^\mu}$ are 4x4 unitary matrices (Dirac-Pauli representation):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \boldsymbol{\sigma^\mu} \\ -\boldsymbol{\sigma^\mu} & 0 \end{pmatrix}$$

every element of $\boldsymbol{\gamma^\mu}$ matrices stands for 2x2 matrix,
1 denotes 2x2 unit matrix,
0 represents 2x2 null matrix

$\boldsymbol{\gamma^\mu}$ are fixed matrices;
$\boldsymbol{\sigma^i}$: Pauli spin matrices (representation of the 1/2 spin operator)

The full version of **Dirac Equation (*DE*)**:

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial z} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \underbrace{\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}}_{\text{bi-spinor}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\boldsymbol{(i\gamma^\mu \partial_\mu - m)\psi = 0}}$$

remember about summation
over repeated indices!

Dirac equation – solutions

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

It’s nice to remember what we wanted to obtain: a wavefunction of a relativistic particle.

Let’s write it now in the form of the plane wave and a Dirac spinor, $u(p^\mu)$:

$$\psi(x^\mu) = u(p^\mu)e^{-i(Et-\vec{p}\cdot\vec{x})}$$

Substituting $\psi(x^\mu)$ into *DE*: $\boldsymbol{(\gamma^\mu p_\mu - m)u(E,\vec{p}) = 0}$

Tutorial!

$$p^\mu = (E,\vec{p})$$

$$p_\mu = (E,-\vec{p})$$

$$x^\mu = (t,\vec{x})$$

$$-ix_\mu p^\mu = -i(Et-\vec{p}\cdot\vec{x})$$

- For a particle at rest, $\vec{p} = \mathbf{0}$, spatial derivatives are 0, *DE* is in the form: $\left(i\gamma^0\frac{\partial}{\partial t} - m\right)\psi = 0$

$$(\gamma^0 E - m)\psi = 0$$

what can be expressed as an eigenvalue problem for the spinors u:

$$\hat{E}u = \begin{pmatrix} m\boldsymbol{I} & 0 \\ 0 & -m\boldsymbol{I} \end{pmatrix}u \qquad E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

- The free-particle wavefunction is: $\psi = u(E,0)e^{-iEt}$

with the eigenspinors:

spin up ↑

spin down ↓

$$\underbrace{u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{E\,=\,m} \qquad \underbrace{u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{E\,=\,-m}$$

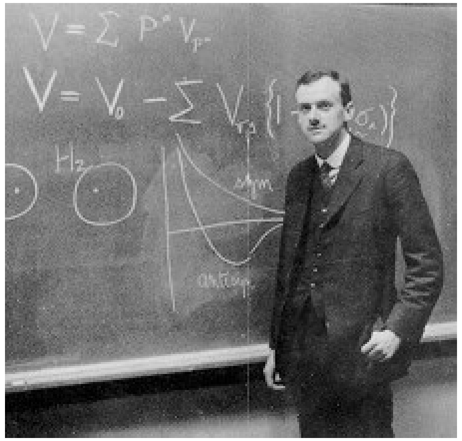
These four states are also eigenvalues of the $\hat{S}_z$ operator, so they represent spin-up and spin-down fermions (why? - later).

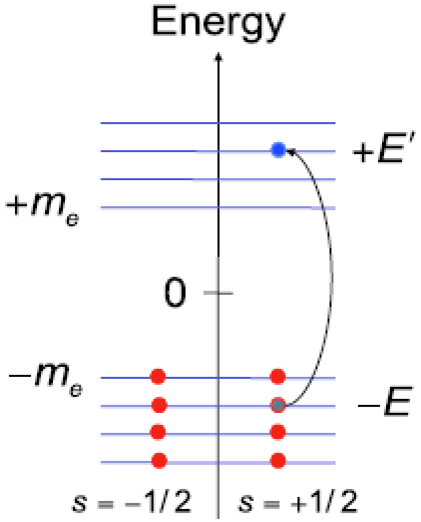
Dirac equation – interpretation I

Four solutions of the Dirac equation for a particle at rest:

$$\psi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imt} \qquad \psi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imt} \qquad \psi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{+imt} \qquad \psi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{+imt}$$

describe two different state of a fermion (↑↓) with $E = m$ and $E = -m$





Dirac's Interpretation:

- Vacuum (fully filled) represents a „sea" of negative energy particles.
- According to Dirac: holes in this „sea" represent antiparticles.
- If energy 2E is provided to the vacuum:

  one electron (negative charge, positive energy) and one hole (positive charge, negative energy are created.
- This picture fails for bosons!

Dirac equation – interpretation II

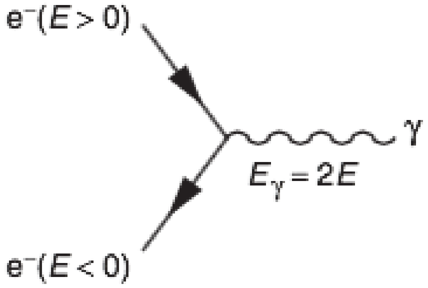
Stückelberg (1941)-Feynman (1948) interpretation of antiparticles*:

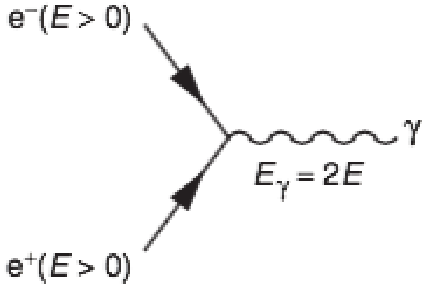
- consider the negative energy solution as *running backwards in time*

  and re-label it as *antiparticle*, with *positive* energy, going forward in time:

$$e^{-i[(-E)(-t)-(-\vec{p})\cdot(-\vec{x})]} = e^{-i[Et-\vec{p}\cdot\vec{x}]}$$

- emission of $E > 0$ antiparticle = absorption of particle $E < 0$





This involves a *CPT* transformation:

- we have flipped Charge (C),
- flipped time (T),
- *and to prevent momentum* from being flipped, must *also* flip the space coordinates (P)

***Feynman–Stueckelberg interpretation** [Wikipedia]
By considering the propagation of the negative energy modes of the electron field backward in time, Ernst Stueckelberg reached a pictorial understanding of the fact that the particle and antiparticle have equal mass **m** and spin **J** but opposite charges **q**. This allowed him to rewrite perturbation theory precisely in the form of diagrams. Richard Feynman later gave an independent systematic derivation of these diagrams from a particle formalism, and they are now called Feynman diagrams. Each line of a diagram represents a particle propagating either backward or forward in time. This technique is the most widespread method of computing amplitudes in quantum field theory today.
Since this picture was first developed by Stueckelberg,[5] and acquired its modern form in Feynman's work,[6] it is called the **Feynman–Stueckelberg interpretation** of antiparticles to honor both scientists.

Dirac equation – general solution

- For a moving particle, $\vec{\boldsymbol{p}} \neq \mathbf{0}$, the Dirac equation can be written using Pauli representation in *DE*

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\psi(x^\mu) = u(p^\mu)e^{-i(Et-\vec{p}\cdot\vec{x})}$$

$$(\gamma^\mu p_\mu - m)u(E,\vec{p}) = 0$$

Tutorial!

using: $(i\gamma^\mu \partial_\mu - m)\psi = 0\,,\ \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$(\gamma^\mu p_\mu - m)(u_A \quad u_B) = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where: $u_A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $u_B = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$

4-component bi-spinor as 2-component vector

- It seems that the equations for vectors u_A and u_B are coupled:

$$(\vec{\sigma}\cdot\vec{p})\,u_B = (E-m)\,u_A$$

$$(\vec{\sigma}\cdot\vec{p})\,u_A = (E+m)\,u_B$$

- Taking the two simplest solutions for u_A and u_B as the orthogonal vectors:

$$u_{A,B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } u_{A,B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Tutorial!

we obtain four orthogonal solutions to the free particle Dirac equation:

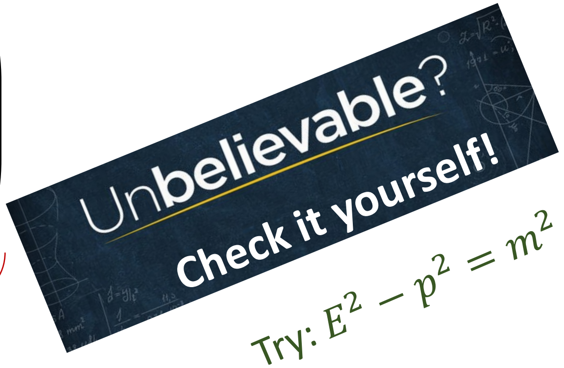
where:

$$\psi_i = u_i(E,\vec{p})e^{-i(Et-\vec{p}\cdot\vec{x})}$$

Dirac equation – solutions for moving particles

… where:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \qquad u_3 = \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$



Try: $E^2 - p^2 = m^2$

electron with energy

$$E = +\sqrt{m^2 + p^2}$$

$$\psi = u_{1,2}(p^\mu)\, e^{-i(Et - \vec{p}\cdot\vec{x})}$$

positron with energy

$$E = -\sqrt{m^2 + p^2}$$

$$\psi = u_{3,4}(p^\mu)\, e^{-i(Et - \vec{p}\cdot\vec{x})} \qquad p^\mu = (E, \vec{p})$$

Now we can take F-S interpretation of antiparticles as particles with positive energy (propagating backwards in time), and change the negative energy solutions $u_{3,4}$ to represent positive antiparticle (positron) spinors $v_{1,2}$:

$$\left.\begin{aligned} v_1(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})} &\equiv u_4(-E,-\vec{p})e^{-i(-Et + \vec{p}\cdot\vec{x})} = u_4(-E,-\vec{p})e^{i(Et - \vec{p}\cdot\vec{x})} \\ v_2(E,\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})} &\equiv u_3(-E,-\vec{p})e^{-i(-Et + \vec{p}\cdot\vec{x})} = u_3(-E,-\vec{p})e^{i(Et - \vec{p}\cdot\vec{x})} \\ E &= +\sqrt{m^2 + p^2} \end{aligned}\right\}$$

reversing the sign of *E* and *p*

The u and v are solutions of:

$$(i\gamma^\mu p_\mu - m)u = 0 \text{ and } (i\gamma^\mu p_\mu + m)v = 0$$

Dirac equation – solutions for moving particles

… where:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

electron with energy

$$E = +\sqrt{m^2 + p^2}$$

$$\psi = u_{1,2}(p^\mu)e^{-i(Et - \vec{p}\cdot\vec{x})}$$

positron with energy

$$E = +\sqrt{m^2 + p^2}$$

$$\psi = v_{1,2}(p^\mu)\, e^{i(Et - \vec{p}\cdot\vec{x})}$$

$$p^\mu = (E, \vec{p})$$

The u and v are solutions of:

$$\boldsymbol{(i\gamma^\mu p_\mu - m)u = 0} \text{ and } \boldsymbol{(i\gamma^\mu p_\mu + m)v = 0}$$

A few conclusions for a break

- There is one remark about the antiparticle solutions:

$$\psi = v(E,\vec{p})\, e^{i(Et-\vec{p}\cdot\vec{x})}$$

- While calculating the operators: $\hat{p} = -i\nabla,\ \widehat{H} = i\frac{\partial}{\partial t}$

  we have: $\widehat{H}\psi = i\frac{\partial v}{\partial t} = -Ev$ and $\hat{p} = -i\nabla\psi = -i\nabla v = -\vec{p}v$ (but we demanded positive energy solutions)

  » that implies that QM operators for energy and momenta of the antiparticle solutions are:

$$\hat{p}^v = i\nabla,\ \widehat{H}^v = -i\frac{\partial}{\partial t}$$

  and if $(E,\vec{p}) \rightarrow (-E,-\vec{p})$ then $\vec{L} = \vec{r}\times\vec{p} \rightarrow -\vec{L}$

  conservation or the total angular momentum requires $[H,\vec{L}+\vec{S}] = 0$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \hat{S}^v \rightarrow -\hat{S}$$

the spin of the antiparticle solution is given by:

$$\hat{S}^v = -\hat{S}$$

- We can draw a conclusion that the appearance of a spin-up fermion must be accompanied by the existence of spin-down antifermion. That's the reason we associated the solution u_1 (spin-up particle) with v_4 (spin-down antiparticle) and u_2 (spin-down particle) with v_3 (spin-up antiparticle)

Summary of Solutions of Dirac Equation

The normalised free particle solutions to the *DE*:

$\psi = u(E,\vec{p})\, e^{-i(Et-\vec{p}\cdot\vec{x})}$ satisfy $(i\gamma^\mu p_\mu - m)u = 0$

with:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

The normalised free antiparticle solutions to the *DE*:

$\psi = u(E,\vec{p})\, e^{i(Et+\vec{p}\cdot\vec{x})}$ satisfy $(i\gamma^\mu p_\mu + m)u = 0$

with:

$$v_1 = \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

- Both particle and antiparticle have positive energy solutions: $E = +\sqrt{m^2+p^2}$.
- Particle and antiparticle have opposite spin: $\hat{S}^v = -\hat{S}$

but now:

The main problem is:
we have no antiparticles....

Antiparticles? Where?

No evidence for the original, “primordial” cosmic antimatter:

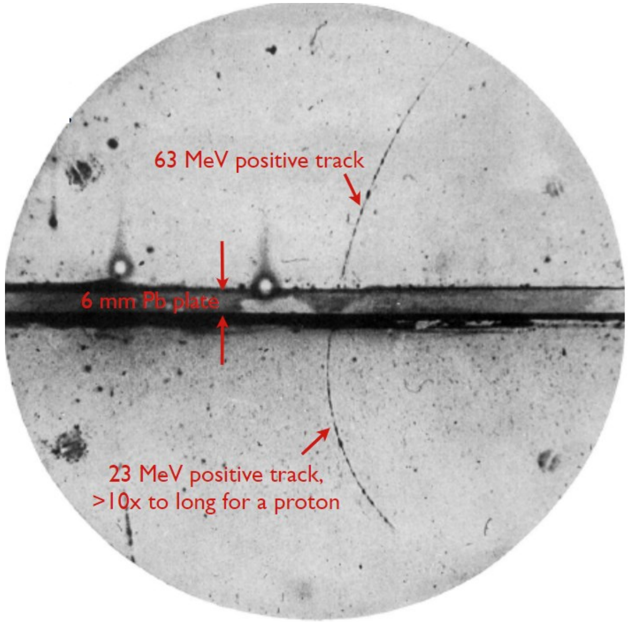
- Absence of anti-nuclei amongst cosmic rays in our galaxy
- Absence of intense γ−ray emission due to annihilation of distant galaxies in collision with antimatter

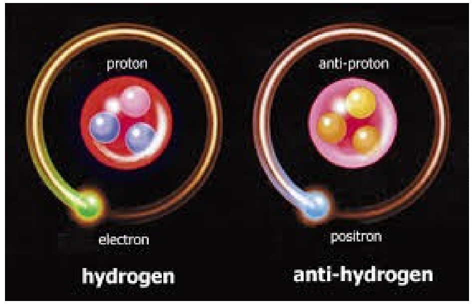
Antiparticles- history

1. **Positron** was discovered in 1933 by Anderson with usage of a Wilson cloud chamber.
2. **Antiproton** in 1955 at the Bevatron (a 6.5 GeV synchrotron in Berkeley)
3. **Antineutron** – 1956.

4. **ANTI-HYDRODEN**:
   Produced in 1995 at the Low Energy Antiproton Ring (LEAR) CERN

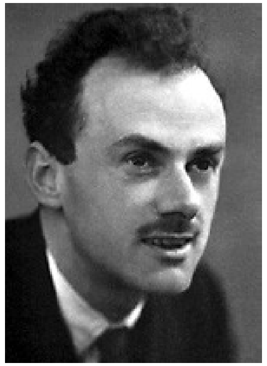
5. Searches of **anti-nuclei** in the space:
   - Alpha Magnetic Spectrometer (AMS-01)
   - Searches for anti-helium in cosmic rays – lots of He found but no anti-He!
   - AMS-02 – extreme flux of positrons detected – consistent with e+e- annihilation, but some increase in high energies of unknown origin.





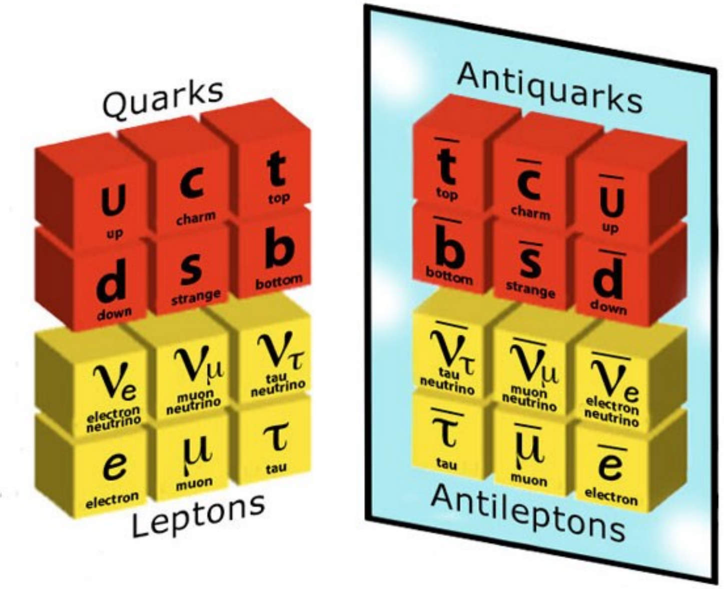
Antiparticles- why?

Dirac's prescience



Concluding words of 1933 Nobel lecture

"If we accept the view of complete symmetry between positive and negative electric charge so far as concerns the fundamental laws of Nature, we must regard it rather as an accident that the Earth (and presumably the whole solar system), contains a preponderance of negative electrons and positive protons. It is quite possible that for some of the stars it is the other way about, these stars being built up mainly of positrons and negative protons. In fact, there may be half the stars of each kind. The two kinds of stars would both show exactly the same spectra, and there would be no way of distinguishing them by present astronomical methods."



The next conclusion

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

- In general u_1, u_2, v_1, v_2 are not eigenstate of the spin operator (can we show that?)
- But if the z-axis is aligned with particle direction: $p_x = p_y = 0, p_z = \pm|p|$, then we get the following Dirac states:

$$u_1 = \begin{pmatrix}1\\ 0\\ \frac{\pm|p|}{E+m}\\ 0\end{pmatrix} \quad u_2 = \begin{pmatrix}0\\ 1\\ 0\\ \frac{\pm|p|}{E+m}\end{pmatrix} \quad v_1 = \begin{pmatrix}0\\ \frac{\pm|p|}{E+m}\\ 0\\ 1\end{pmatrix} \quad v_2 = \begin{pmatrix}\frac{\pm|p|}{E+m}\\ 0\\ 1\\ 0\end{pmatrix}$$

which are eigenstates of the $\hat{S}_z$ operator:

$$S_z u_1 = +\frac{1}{2}u_1\,; \quad S_z^v v_1 = -S_z v_1 = +\frac{1}{2}v_1$$

$$S_z u_2 = -\frac{1}{2}u_2\,; \quad S_z^v v_2 = -S_z v_2 = -\frac{1}{2}v_2$$

- Solutions of the *DE* are spin eigenstates only if particle is travelling along the z-axis.
- So let's go further – define an operator that is the projection of the spin along the direction of motion…

Welcome to the helicity world!

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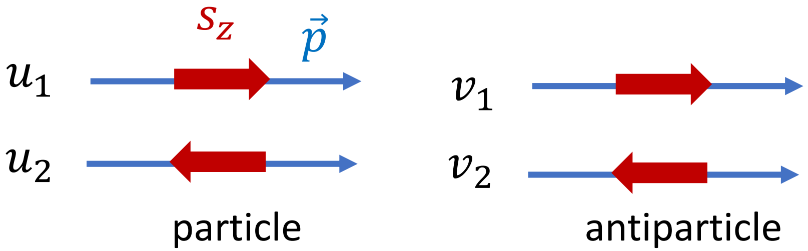
Spin and helicity

- The two different solutions of *DE* of the fermions and antifermions correspond to two possible spin states.

If we choose the „z” axis along the momentum $\vec{p}$:



we can define **helicity operator**: $\hat{\boldsymbol{h}} = \frac{\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{p}}}{2\,\boldsymbol{p}} = \frac{\boldsymbol{1}}{\boldsymbol{2p}}\begin{pmatrix}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\end{pmatrix}$

as a projection of spin along the particle’s direction of flight.

if $h = +1$ particle is right-handed

if $h = -1$ particle is left-handed

we can express the Dirac spinors as a right- and left- handed helicity spinors.

But from the experiment:

- massless fermions are purely left-handed (only u_2);
- massless antifermions are purely right-handed (only v_1) .

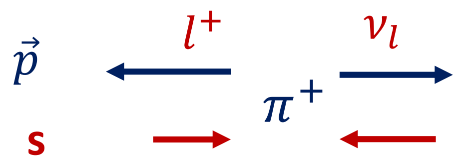
it seems that we need another representation of the Dirac spinors

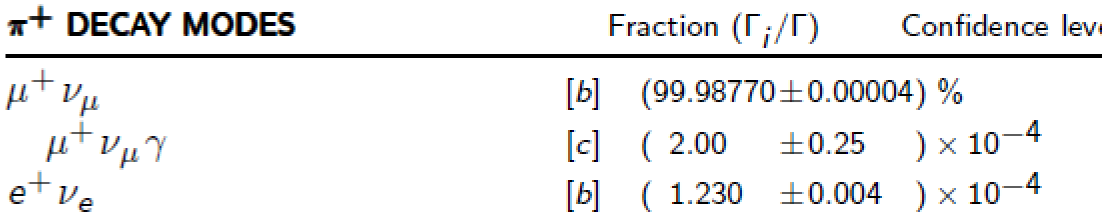
- In addition helicity is not Lorentz invariant

Spin structure of weak interactions

Example: Only the left bi-spinor describes fermions (in ultrarelativistic limit) emitted in charged current weak interaction.

Charged pion decay:





π^+ DECAY MODES		Fraction (Γ_i/Γ)	Confidence lev
$\mu^+\nu_\mu$	[b]	(99.98770 ± 0.00004) %	
$\mu^+\nu_\mu\gamma$	[c]	$(2.00 \pm0.25) \times 10^{-4}$	
$e^+\nu_e$	[b]	$(1.230 \pm0.004) \times 10^{-4}$	

- momentum of lepton and neutrino must be in opposite directions,
- spin – also (to satisfy the angular momentum conservation,
- neutrino is left-handed (assuming zero mass) and it followed that lepton must be also left-handed,
- muon is non-relativistic and can both left- or right- handed
- positron is relativistic, only right state is allowed

The Weyl (1929) representation of spinors describes massless (or $E \gg m$) particles
It internally included the parity violation, what was hardly to take in that time.
QED, which is fully symmetric under P operation, is described in Dirac representation.
The description of weak interaction requires additional rules.

… let’s start with the last definition today: quantity called „chirality”…

Chirality

- It is useful to have operators which are Lorentz invariant, not necessarily commuting with Hamiltonian.
- Operator revealing the **chirality:** $\gamma^5 \equiv i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (Lorentz invariant).
- For the massless particles, or in the limit $E \gg m$, the chirality is identical to the helicity.
- The eigenstates of the chirality operator are:

$$\left.\begin{array}{ll} \gamma^5 u_R = +u_R & \gamma^5 u_L = -u_L \\ \gamma^5 v_R = -v_R & \gamma^5 v_L = +v_L \end{array}\right\} \text{right- and left-handed chiral states}$$

γ^5 is a 4x4 matrix,

$(\gamma^5)^2 = 1$

$\gamma^{5\dagger} = \gamma^5$

$\{\gamma^5, \gamma^\mu\} = 0$

Tutorial!

- Chirality is also defined in terms of the chiral projection operators:

$$P_L = \frac{1}{2}(1 - \gamma_5); \qquad P_R = \frac{1}{2}(1 + \gamma_5)$$

such that P_L projects out the left-handed particles chiral states and right-handed chiral anti-particle states: P_R projects out the right-handed particles chiral states and left-handed chiral anti-particle states

(remember that the direction of momentum was reversed in case of antiparticles.

$$\begin{array}{c} P_L u_R = 0 \\ P_L u_L = u_L \\ P_L v_R = v_R \\ P_L v_L = 0 \end{array}$$

$$\left.\begin{array}{l} P_L\,P_R = P_R\,P_L = 0 \\ P_L + P_R = 1 \end{array}\right\}$$

Any spinor can be written in terms of its left and right-handed chiral states:

$$\boxed{\psi = (P_L + P_R)\psi = P_L\psi + P_R\psi = \psi_L + \psi_R}$$

- Chirality plays an important role in the Standard Model probability current calculation…. Only certain combinations of chiral eigenstates contribute to the interaction… (soon on this course).

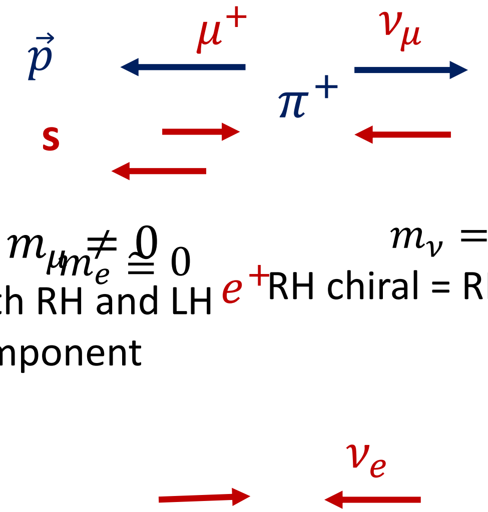
Chiral and helicity states

- Dirac solutions ≡ fermion and antifermion states $\{u_1, u_2, v_1, v_2\}$
- Helicity ≡ states projection of spin into momentum direction $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$ or $\{u_R, u_L, v_R, v_L\}$
- Chiral states ≡ eigenstates of the γ^5 matrix. $\{u_R, u_L, v_R, v_L\}$

A few remaks:

- Velocity of a particle defines a direction in any reference frame.
- The states of definite helicity are the eigenstate of the third component of the spin in that direction.
- Helicity is not Lorentz invariant unless $m = 0$.
- If the particle is massive one can always find a reference frame in which particle travel in opposite direction, thus change helicity.
- Helicity is a property of a particle.
- Chirality is a property of 4-component spinor.
- Only if $m = 0$ or $E \gg m$ RH chiral and helicity states are identical.

Tutorial!



$m_\mu \neq 0$ $m_e \neq 0$
both RH and LH component

$m_\nu = 0$
RH chiral = RH helicity

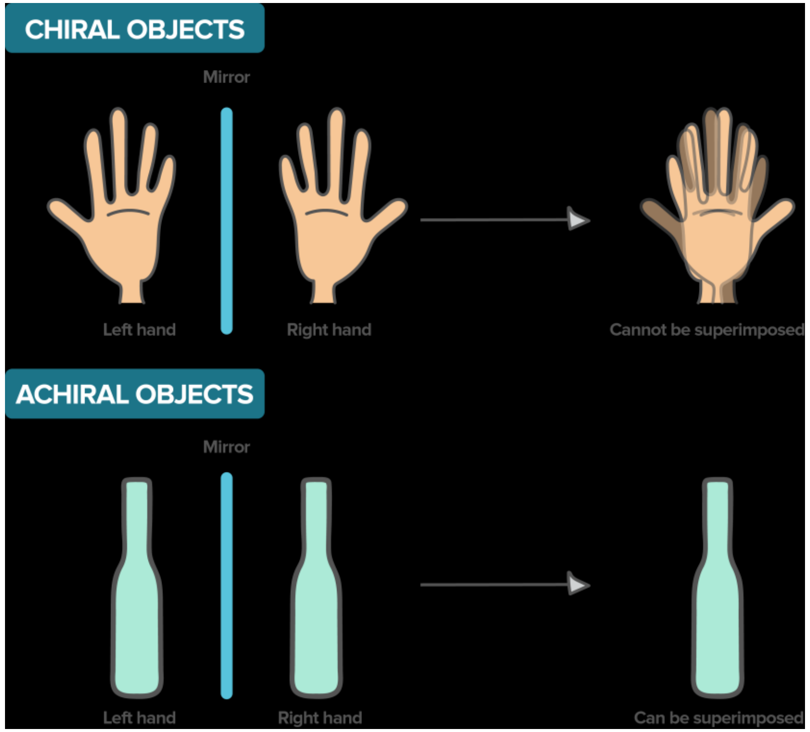
Chirality in pictures*

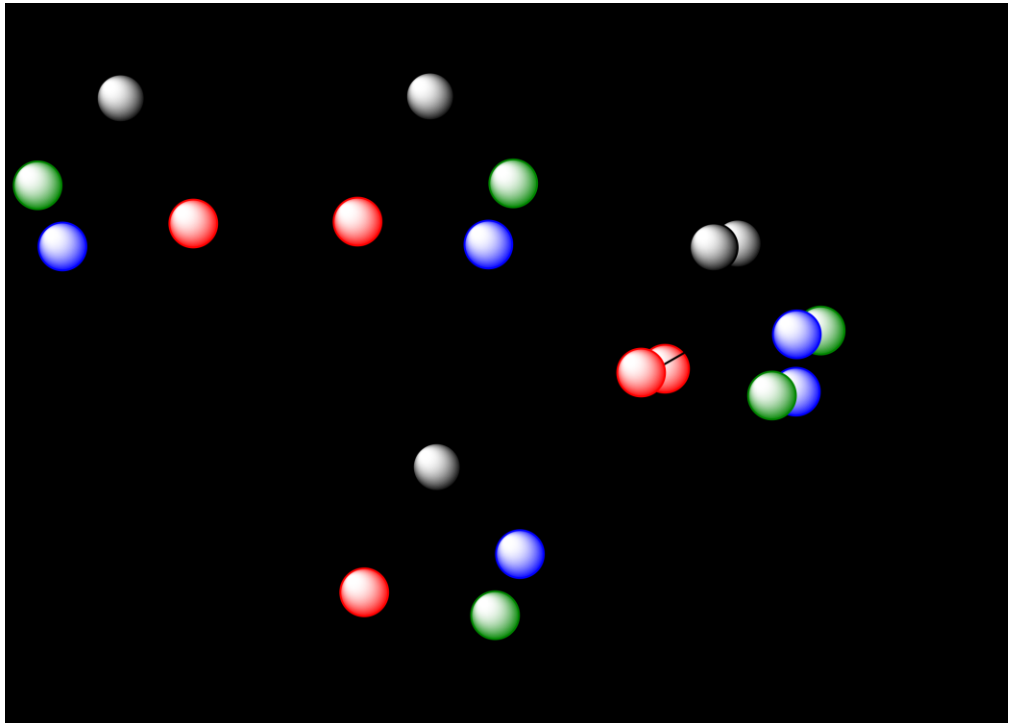
What is chirality?

Chirality is derived from the Greek word χειρ (kheir) that stands for "hand". An object is said to be chiral if the object and its mirror image are non-superimposable, just like our right and left hand. Now you must be wondering what we mean by ‘non-superimposable’. When the mirror image of the object is placed over the original object and they do not overlap, as shown in the figure below, then the object and its image are said to be non-superimposable.

Molecular chirality was discovered by Louis Pasteur back in 1848, when he successfully separated the two isomers of sodium ammonium tartarate. He observed that the two isomeric crystals were non-superimposable mirror images of each other, they had the same physical properties, but differed in their ability to rotate plane polarized light. This property was termed as optical activity.

https://www.khanacademy.org/test-prep/mcat/chemical-processes/stereochemistry/a/chiral-drugs





Summary

- Dirac Equation is a marriage of Quantum Mechanics and Special Relativity.
- It describes the wave function of fermions and antifermions as spinors.
- This picture is incomplete: it is experimentally shown that particle/antiparticle symmetry is broken.
- The chiral representation was introduced to enhance the differences between weak interaction in massless fermions.